

Toward an Understanding of Mathematical Procedures

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Research suggests that algebraic skill is intricately connected to understanding. Without understanding, algebraic performance is fraught with error and skills are quickly forgotten. In this article, we review one model of algebraic understanding and offer five easy-to-use classroom questions that are based upon this model. These questions represent a concrete approach to achieving the level of student understanding that is advocated in the Common Core Mathematics Standards. Through the use of these questions and related “understanding” investigations and assessments, students can attain both computational fluency and conceptual understanding. In-depth examples of the application of these questions to a linear equation task are provided, as are sample student investigations that can be used to promote an understanding of mathematical procedures.

Introduction

There is growing awareness that students’ ability to use mathematical procedures is directly related to their understanding of these procedures. The *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010), for instance, suggest that an understanding of the conceptual underpinnings of one task (e.g., the use of the distributive law to expand $(a + b)(x + y)$) facilitates the learning of more complex tasks. According to the *Common Core Standards for Mathematics* (CCSM), procedural fluency and conceptual understanding are not mutually exclusive aspects of students’ knowledge, but should be developed concurrently. This dual emphasis on concepts and procedures mirrors the recommendations of the *Principles and Standards for School Mathematics* (NCTM, 2000) which contends that in addition to computational skill, students need to “understand the concepts of algebra, the structures and principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations” (PSSM, p. 37).

Although neither the CCSM nor the NCTM *Standards* are research documents, research on the teaching and learning of mathematics supports the combined emphasis on concepts and procedures. When classroom instruction focuses on understanding mathematics, procedures are executed intelligently and with fewer errors (Rittle-Johnson & Koedinger, 2002; Star & Siefert, 2002); knowledge that is understood lasts longer and can be widely applied (Carpenter & Lehrer, 1999); new knowledge is easier to learn (Hiebert & Carpenter, 1992); and knowledge that is forgotten can easily be recreated (Carpenter & Lehrer, 1999). These benefits may explain the recent findings of Rakes and his colleagues (2010) who report that instructional reforms focusing on conceptual understanding result in higher student achievement.

Despite calls for instruction that promotes both understanding and fluency (and the plethora of research supporting this approach), the teaching of mathematics in U.S. classrooms has not significantly changed. Although teachers often cite the lack of time as a barrier to this transition, the vague nature of the task

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itself is also an impediment. In particular, we have found that many teachers desire to promote conceptual understanding but have yet to find explicit and effective strategies for doing so. In *Navigating through Algebra in Grades 9-12*, Burke and his colleagues (2001) define mathematical understanding in terms of six literacies. In our efforts to promote mathematical understanding in the classroom, we have found Burke's literacies to be both descriptive and proscriptive. In this article, we adapt Burke's literacies into easy-to-use and widely applicable classroom questions. By applying these questions to a common procedural task (solving linear equations), we illustrate how one can shift the focus of instruction from simply solving problems to solving problems with understanding.

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A Framework for Understanding Procedures

One often equates success in algebra with an ability to manipulate algebraic symbols. Based upon extensive clinical interviews with algebra students, however, Burke and his colleagues (2001) found that successful students' understanding of algebra is actually quite varied. Some are able to manipulate symbols, but possess little understanding of the meaning of these actions. Lacking understanding, these students cannot apply their knowledge to novel situations, nor do they retain their computational abilities. The most successful students, on the other hand, possess much deeper and connected understandings of algebra. On the basis of these observations, Burke identified six literacies that represent "ideal" understanding of mathematical procedures:

1. The student understands the overall goal of the procedure and can predict or estimate the outcome.
2. The student understands how to carry out the procedure and knows alternative

methods and representations of the procedure.

3. The student understands and can communicate to others why the procedure is effective and valid.
4. The student understands how to evaluate the results of the procedure by invoking connections with a context, alternative procedures, or other mathematical ideas.
5. The student understands and uses mathematical reasoning to assess the relative efficiency and accuracy of the procedure as compared to alternative methods.
6. The student understands why the procedure empowers her or him as a problem solver.

According to this framework, the traditional focus on computation (Literacy 2) remains an important component of procedural understanding. If students can only check their work by repeating the same sequence of steps, however, their understanding is limited. Students that understand procedures can check their work by estimating the answer (Literacy 1) and using alternative solution strategies (Literacy 4). They can also compare the relative efficiency of alternative strategies (Literacy 5), clearly explain why each step of their solution is valid (Literacy 3), and understand the connections between each procedure and other mathematical (and non-mathematical) tasks (Literacy 6). According to Burke and his colleagues, mathematical understanding and long-term procedural competence emerge from experiences that enable students to develop all six literacies.

In the Classroom

For several years, we have incorporated Burke's algebraic literacies into our own teaching and professional development with teachers. In both settings, this has involved revising lessons to include examples,

explorations, and assessments that explicitly address all six literacies. To achieve the goal of procedural understanding, however, we have found that these revisions must extend beyond our actions as teachers and empower students to seek understanding on their own. The following five questions, illustrated within the context of a common procedural task, are easily incorporated into most lessons and effectively promote classroom discourse. More importantly, when asked regularly, these questions promote habits of mind that lead to procedural literacy.

1. What sort of answer should I expect?

Estimation is common in elementary classrooms, but is rarely utilized in later years. Yet, asking students to estimate answers before enacting a solution strategy can deepen their understanding of the solution process. When asked to estimate the answer to $2x + 5 = x + 17$, for instance, most students focus on the nature of the solution. Specifically, the solution is a number that, when substituted for x , yields the same value for the left-hand and right-hand side of the equation. If posed in a slightly different manner, however, estimation questions can elicit more specific answers. Before solving the equation, for instance, ask students whether the solution to the equation is positive or negative. In response to this question, some of our students used a guess-and-check approach. When $x = 0$, the value of the left-hand side of the equation is less than the right-hand side. On the other hand, the value of the left-hand side is greater than that of the right-hand side when $x = 20$. For some positive number between 0 and 20, therefore, the left and right sides of the equation are equal.

Solving systems of equations offers another opportunity to effectively utilize

estimation. The system of equations $y = 0.5x - 6$ and $2x - 4y = 6$, for instance, has no solutions. Before solving this system, however, ask students to reflect on the potential intersections of two linear equations (see Figure 1). The fact that this system results in no solutions, therefore, is one of the “expected” results. As with estimating solutions to mathematical problems, estimating the results of procedural tasks before completing them allows students to assess whether their solution “makes sense.”

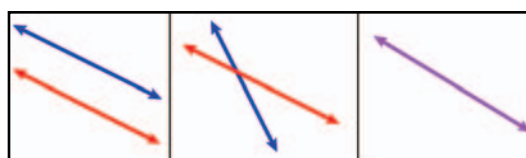


Fig 1 Possible intersections of two linear equations - (Left) No solutions; (Middle) One solution; (Right) Infinite solutions

2. Why does the procedure work?

Justifications convince others of the validity of a particular statement. To justify their work in algebra, therefore, students should focus on the mathematical underpinnings of the procedure and communicating these underpinnings to others. Although students should ultimately cite appropriate field properties and properties of equality, informal arguments that mimic the equation-solving process are appropriate first steps and can be quite beneficial. For instance, to justify the addition of -5 to each side of $2x + 5 = x + 17$, the student may simply explain that he or she is “collecting all constants on one side of the equation.” Note that this explanation doesn’t address the underlying mathematical rationale (additive property of equality) but does establish a viable first step and the rationale for this step. Through the development of justifications, students establish meaningful routines for all procedural tasks.

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3. Is this the best procedure to use?

Questions about the “best” procedure arise when students solve a problem using some procedure and then review the efficiency of this procedure as compared to other approaches. After solving $2x + 5 = x + 17$ via the traditional algorithm, for instance, we recently asked one class to solve the problem in another way. In response to this question, several students created tables of values for the right- and left- hand sides of the equation for various values of x , as in Table 1. In this table, note that an x -value of 12 yields the same value (29) for both the right- and left-hand expressions. The students’ use of tables, therefore, highlights the overall goal of the procedure.

x	$2x + 5$	$x + 17$
9	23	26
10	25	27
11	27	28
12	29	29

Table 1 Tabular solution to $2x + 5 = x + 17$

As a follow-up, we asked students to consider the tabular approach to the equation $8x + 5 = x + 17$. As Table 2

x	$8x + 5$	$x + 17$
0	5	17
1	13	18
2	21	19
3	29	20

Table 2 Tabular approach to $8x + 5 = x + 17$

illustrates, the solution is between 1 and 2. One can improve the estimate of the solution by examining successively narrower intervals (e.g., tenths, hundredths), but the repeating nature of the decimal form of the solution $\frac{12}{7}$ implies that the iterative process will never end. Tables and traditional algorithms both yield solutions, but the only latter approach is guaranteed to yield an exact solution.

4. How do I know my answer is correct?

Students gain mathematical power when they can determine whether or not their answers are correct. Yet, the verification of one’s work is rarely the focus of instruction. In response to this question, encourage students to check their work using alternative solution strategies. After solving $2x + 5 = x + 17$ using traditional algorithms, for instance,

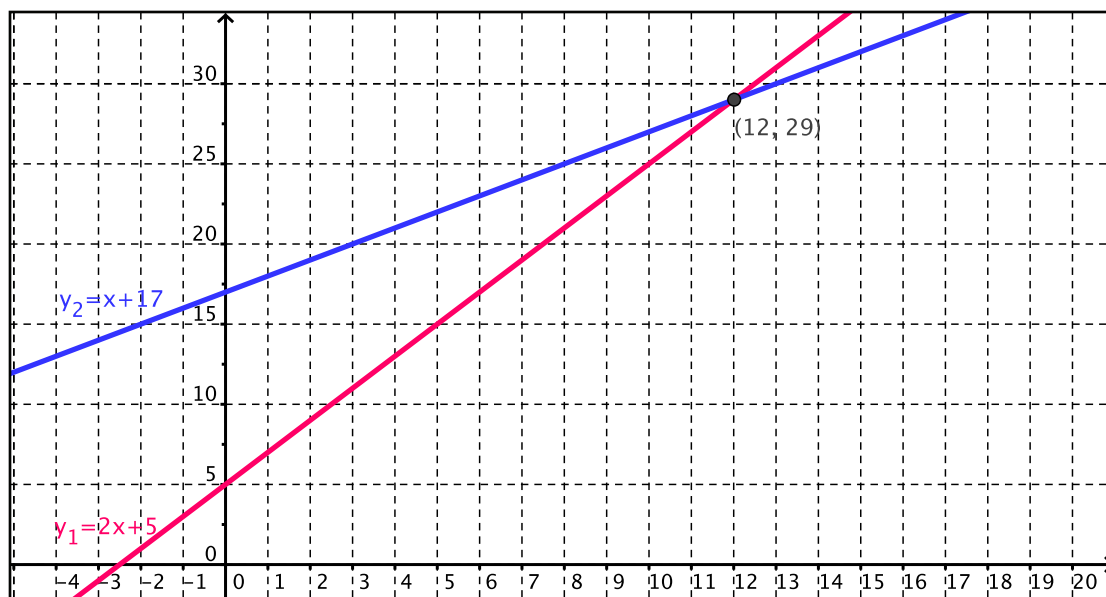


Fig 2 Graphical representation of the system $y_1 = 2x + 5$ and $y_2 = x + 17$

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students can check their work by substituting the solution ($x = 12$) into each side of the equation. If $x = 12$, $2x + 5 = 2(12) + 5 = 25 + 5 = 29$.

Likewise, if $x = 12$, then $x + 17 = 12 + 17 = 29$. Since $x = 12$ yields a true statement ($29 = 29$), 12 is one solution to the equation.

An alternative response to this question, and one that strengthens the connection between graphical and symbolic representations of algebra, is to view each side of the equation as a unique equation – and then graph each equation. If $y_1 = 2x + 5$ and $y_2 = x + 17$, for instance, then the solution to the equation is the x -value of the solution to the system of equations $y_1 = 2x + 5$ and $y_2 = x + 17$. As Figure 2 illustrates, y_1 and y_2 are equal when $x = 12$.

Asking students to verify their answers can be particularly revealing. An elementary colleague recently asked his students to subtract 37 from 46. Having learned the multi-digit subtraction procedure, most of his students successfully found the difference of nine. When asked whether this solution was correct, however, students' limited understandings of the subtraction procedure became apparent. After attempting a variety of strategies (e.g., adding 37 and 46) and achieving no success, one student finally exclaimed, "I don't have any idea if my answer is right." Subsequently, the teacher discussed number line representations of the difference and the definition of subtraction (i.e., $46 - 37 = 9$ iff $37 + 9 = 46$). In general, the quest for verification can reveal students' thinking, deepen their understanding of procedures, and ultimately provide them with additional tools to check the accuracy of their own work.

5. What can I use this procedure to do?

Harel (1998) states that students are

more likely to learn mathematics when they have an intellectual need to do so. Classroom instruction, therefore, should explicitly seek to connect the learning of procedures with students' mathematical needs and interests. For instance, with regard to linear equations students should understand that linear equations appear in many contexts, both within and outside of mathematics.

Within mathematics, the ability to solve linear equations facilitates the solving of many non-linear equations (e.g., $(x - 2)(2x + 1) = 0$). Likewise, all analytic approaches to systems of linear equations reduce the system to a single, one-variable linear equation. Outside of mathematics, linear equations are instrumental in many modeling situations, such as the classic forensic activity relating height and foot size (Carspecken, et al., 2003). The data for this activity typically yield a linear equation relating the two variables, as in Figure 3. For these data, the linear regression model relating foot size and height is $y = 1.76x + 42.16$, where x and y represent foot size and height (in inches), respectively. To find the foot size that corresponds to a height of 60 inches, students must substitute 60 for y and solve the resulting equation. While learning that mathematics is applicable to the real-world will not motivate every student, it does establish that the reach of mathematics extends beyond the four walls of the classroom.

Conclusions

Burke's procedural literacies – and the easy-to-use classroom questions that we present in this article – embody many of the *Common Core Standards for Mathematical Practice*, develop productive habits of mind, and promote conceptual understanding. Through our own teaching and field work with classroom teachers, we have found that

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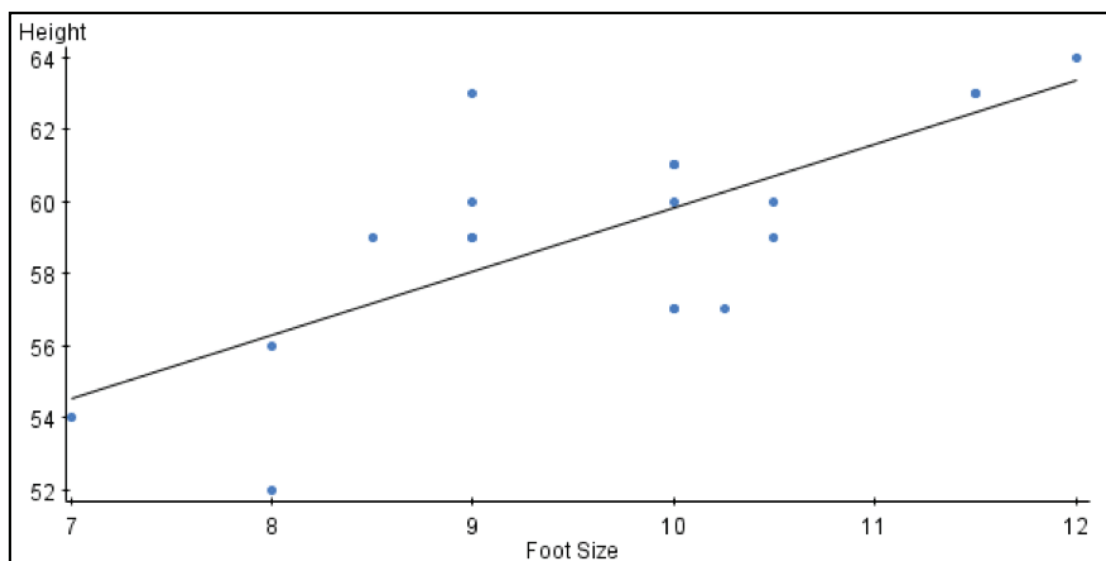


Fig 3 A linear model of the foot size-height relationship

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the key to successful implementation is an explicit focus on Burke's literacies through discourse and investigative activities. One can approach understanding iteratively, for instance, beginning with question 1. Prior to solving procedural problems, ask students to estimate the solution. This strategy elevates students' thinking beyond the steps of solving the problem and allows them to check the reasonableness of their work. Once estimation becomes habitual, encourage students to develop explanations of their solution strategies, examine alternative strategies, and use these strategies to verify solutions (questions 2, 3 and 4). At all times, promote engagement and learning by establishing meaningful connections between students' mathematical work and interests (via question 5). Of course, attempting to address each of the five understanding questions with each classroom task is impractical. However, it is feasible to pose some of these questions or assign understanding investigations in most any lesson (see the Appendix for sample investigations).

Homework assignments offer another opportunity to deepen students' understanding of mathematical procedures. In one assessment of the relative value of procedural and conceptual

practice, Hasenbank (2006) altered the assignments of three intermediate algebra classes and compared the results with three "traditional" classes. The traditional classes completed approximately 18 procedural practice problems after each lesson, whereas the experimental classes completed 12-14 procedural problems and 2-3 conceptual questions (also based upon Burke's model). At the end of the experiment, the classes receiving conceptual questions performed as well as or better than those receiving only procedural practice – despite lower entry-level skills and fewer procedural homework questions. Moreover, experimental students scored much higher on end-of-course conceptual understanding questions. As with classroom lessons and explorations, season your assessments with conceptual questions and homework can become a vehicle for developing computational fluency *and* conceptual understanding.

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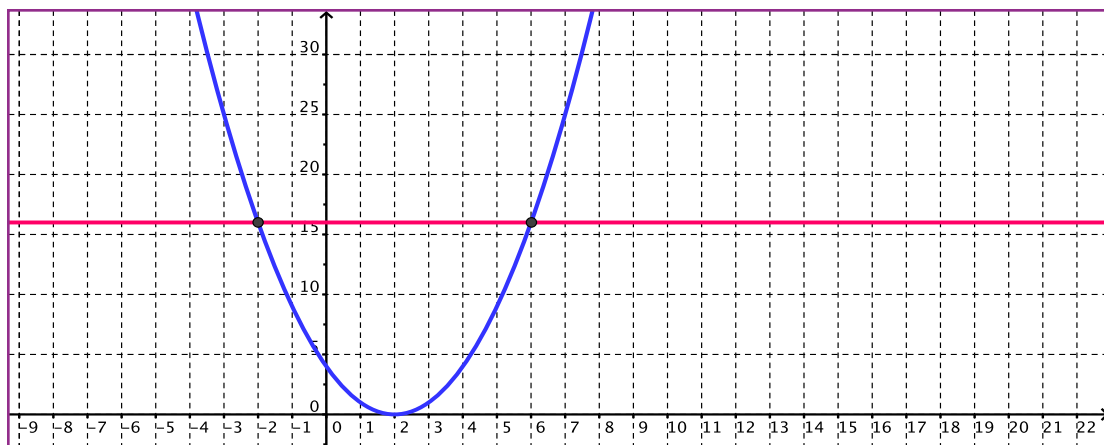
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Appendix: Sample Understanding Investigations

- Samuel claims that the $y = -6$ is the solution to the equation $\frac{1}{2}x - 12 = 3x + 4$. Is he correct?
- Solve $(3x - 4)^2 = 16$ in two ways: (a) expanding the left-hand expression, grouping like terms, and factoring the result, and (b) using the square root method. Which method do you prefer and why?
- If a quadratic expression $(2x^2 + 2x - 12)$ is divided by a linear expression $(x - 2)$, will the quotient be (a) a linear expression, (b) a quadratic expression, (c) a cubic expression? Explain your reasoning.
- The cross-multiplication procedure states that if $\frac{y+3}{8} = \frac{2y-9}{15}$, then $15(y + 3) = 8(2y - 9)$. The steps in the table below provide “proof” of this statement. Provide an explanation for each step in this process. How does each step follow from the step above it?
- The diagram below depicts the graphs of $y = 16$ and $(x - 2)^2$. Explain how these graphs can be used to solve the equation $(x - 2)^2 = 16$.

Steps	Reasons
1: $\frac{y+3}{8} = \frac{2y-9}{15}$	Given
2: $15 \times \frac{y+3}{8} = 15 \times \frac{2y-9}{15}$	
3: $\frac{15(y+3)}{8} = \frac{15(2y-9)}{15}$	
4: $\frac{15(y+3)}{8} = \frac{(2y-9)}{1}$	
5: $\frac{15(y+3)}{8} = 2y - 9$	
6: $8 \times \frac{15(y+3)}{8} = 8(2y - 9)$	
7: $\frac{(8)(15)(y+3)}{8} = 8(2y - 9)$	
8: $15(y + 3) = 8(2y - 9)$	



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